Stability of solitons in nonlinear fiber couplers with two orthogonal polarizations

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In T. Lakoba, D. Kaup, and B. Malomed [Phys. Rev. E **55**, 6107 (1997)], the stationary solitons of the nonlinear directional coupler (NLDC) with two polarizations in each core were studied and detailed by means of the variational method. In the present work, we show how one can analytically determine the stability of all the various solitons found in that previous work in the limit of large soliton energy. We emphasize that our analysis is *not* based on the variational approximations for the solitons, but rather on their asymptotically exact forms in the limit of large energy. We find that in all but one case, the stability of those solitons in this model, which are analogs of any soliton of the NLDC, is the same as that of the corresponding NLDC soliton. We also discuss how our results, valid for large soliton energies, can be extended to finite values of energy. [S1063-651X(97)05910-2]

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I. INTRODUCTION

Dynamics of solitary waves (solitons, for brevity) in nonlinear optical fibers supporting propagation of two coupled modes has recently been a subject of intensive studies [1–10]. Such a dynamics is quite rich due to the fact that in the two-component models, there usually exist more than one stationary soliton state, which can be either stable or unstable depending both on its internal parameters and on the parameters of the model. Thus the issue of linear stability of solitons is crucial for studying their dynamics.

In [1], stability of solitons in two linearly coupled, single-mode fibers [this model is also referred to in the literature as the nonlinear directional coupler (NLDC)] was studied numerically. The NLDC can have, depending on the value of a certain soliton parameter, up to three types of solitons. (Below we will refer only to ''single-humped'' solitons, the reason being that solitons whose profiles have more than one maximum have always been found to be unstable.) Two of them, symmetric and antisymmetric, were shown to be stable only for sufficiently low values of the soliton's total energy. The solitons of the third type (asymmetric) were found to be stable for all energy values larger than a certain threshold. Even before paper [1], Wright *et al.* [2] had studied the stability of the symmetric and antisymmetric solitons (the asymmetric soliton was not known at that time) in a more general model than the NLDC, which also included the nonlinear coupling between the modes. In Refs. [3,4], the (mostly numerical) studies were concerned with the stability of phase-locked solitons in a *weakly* birefringent fiber. Such a fiber is known to have two linear eigenmodes, called fast and slow modes, in accordance with the phase velocity they have relative to one another. There have been three different types of solitons found in such a model [4]. It was shown in Refs. [3,4] that solitons of the first type, whose polarization is aligned along the slower eigenmode of the fiber, are always (i.e., for all energies) stable, while the solitons of the second type, with polarization aligned along the fast eigenmode, are unstable for almost all values of their energy. The solitons of the third type, which were found in [4] and which have nonvanishing components in both fast and slow modes, were shown to be weakly unstable. To conclude this very brief overview of stability of two-component solitons in optical fibers with Kerr nonlinearity, we mention that solitons in *strongly* birefringent fibers were shown, both analytically [5–7] and numerically [8,9], to be linearly stable. Nevertheless, the dynamics of a near-soliton initial pulse in such a system can also be quite nontrivial, since it has recently been shown that it can have (depending on the value of the parameter of cross-nonlinearity) a long-term internal oscillating mode [10].

Recently, we have considered in Ref. [11] the model of two linearly coupled optical fibers, with each fiber supporting propagation of two orthogonal eigenmodes with distinctly different phase velocities. This model is a natural generalization of the NLDC model, mentioned above. On the other hand, when the fibers in our model are taken far apart from each other, and thus the linear coupling between them is eliminated, then one obtains the equations of pulse propagation in a single, strongly birefringent fiber [12], which have also been thoroughly studied (see, e.g., [10], and references therein). The equations considered in [11] have the following form:

$$
\begin{aligned}
&iu_{1,z}+\tfrac{1}{2}u_{1,\tau\tau}+u_1(|u_1|^2+\beta|v_1|^2)+\kappa u_2=0,\\
&iv_{1,z}+\tfrac{1}{2}v_{1,\tau\tau}+v_1(|v_1|^2+\beta|u_1|^2)+\kappa v_2=0,\\
&iu_{2,z}+\tfrac{1}{2}u_{2,\tau\tau}+u_2(|u_2|^2+\beta|v_2|^2)+\kappa u_1=0,\\
&iv_{2,z}+\tfrac{1}{2}v_{2,\tau\tau}+v_2(|v_2|^2+\beta|u_2|^2)+\kappa v_1=0,
\end{aligned}
\tag{1.1}
$$

where we have used the standard undimensionalized variables $u_{1,2}$ and $v_{1,2}$ for the envelopes of the electric field, and z and τ for the distance along the fiber and the time in the pulse's reference frame, respectively (cf., e.g., [13]). Following [11], we will refer to Eqs. (1.1) as the ''dual core, dual polarization'' model (DCDP). Let us note that the value of the linear coupling constant κ in Eq. (1.1) can be scaled to any nonzero value. In [11], we only considered the cases when $\beta=2/3$ and $\beta=2$, which correspond, respectively, to propagation of linearly and circularly polarized eigenmodes in each fiber. Any values of β between 2/3 and 2 will correspond to elliptically polarized eigenmodes [12,13], and qualitative predictions about the solitons of Eqs. (1.1) with $2/3<\beta<2$ can be made on the basis of the information obtained for the limiting cases of $\beta=2/3$ and $\beta=2$.

In [11], we studied the problem of the existence of stationary solitons of the DCDP with the variational method. We approximated the soliton's components by Gaussian trial functions:

$$u_{1,2}=A_{1,2}e^{-a^2\tau^2/2}e^{ipz}, \quad v_{1,2}=B_{1,2}e^{-b^2\tau^2/2}e^{iqz}, \quad (1.2)$$

and for the stationary amplitudes $A_{1,2}$, $B_{1,2}$ and widths a, b, we derived a system of nonlinear algebraic equations, in which p and q played the role of control parameters. Then from those equations, we found the boundaries of the regions of existence for all types of solitons of Eqs. (1.1) and also numerically calculated the typical profiles of the solitons.

In this work we present the stability analysis for all the types of solitons found in [11], in the limit when the soliton's energy, defined as

$$E=E_u+E_v, \tag{1.3}$$
$$E_u=\int_{-\infty}^{\infty}(|u_1|^2+|u_2|^2)d\tau, \quad E_v=\int_{-\infty}^{\infty}(|v_1|^2+|v_2|^2)d\tau,$$

is large. The principal idea that allows one to perform such an analysis is the following. One can show that the limit $E\gg1$, κ fixed in Eqs. (1.1) is equivalent to the limit E fixed, $\kappa\ll1$, i.e., the small coupling limit. Then by means of this scaling, one can consider the linear coupling in Eq. (1.1) as a small perturbation for a two-component soliton in a single, strongly birefringent fiber. Thus the original problem of the stability of solitons of the DCDP in the limit $E\gg1$, κ fixed is reduced to the problem of stability of solitons in the strongly birefringent fiber with such a perturbation, for which the theory can be constructed along the standard lines. Let us emphasize that our analysis pertains to the *exact*, asymptotic (for $E\gg1$) solutions of Eqs. (1.1) and *not* to their variational approximations obtained in [11]. The role of the variational method was to determine which *particular* configuration, out of all possible ones, of the soliton's components [see Eqs. (2.12)–(2.14) below] can be realized in the limit of $E\gg1$.

The remainder of the paper is organized as follows. In Sec. II we will review the results obtained in [11] for the solitons of the DCDP, as well as some relevant results for earlier models. The details of our stability analysis will be presented in Secs. III and IV, with the generic case being treated in Sec. III and a degenerate one in Sec. IV. In Sec. V we summarize and discuss the results obtained; in particular, we discuss how our stability results can be continued from $p,q\gg1$ to the finite p and q. We also formulate open questions regarding the stability of solitons of the DCDP. It should be noted that the main results of our analysis were already announced in [11].

II. REVIEW OF RESULTS ON THREE EARLIER MODELS

Since Eqs. (1.1) are a generalization of both the NLDC model and the model of a single birefringent fiber with two orthogonal polarizations, it is natural to start with a review of the well-known results for these two models.

The NLDC is described by two nonlinear-Schrödinger (NLS) type equations:

$$\begin{aligned} &iu_{1,z}+\tfrac{1}{2}u_{1,\tau\tau}+u_1|u_1|^2+u_2=0,\\ &iu_{2,z}+\tfrac{1}{2}u_{2,\tau\tau}+u_2|u_2|^2+u_1=0, \end{aligned} \tag{2.1}$$

where the linear coupling constant has been set equal to unity. The solitons of Eqs. (2.1) are sought in the form

$$u_n(z,\tau)=e^{ipz}u_n(\tau), \quad n=1,2 \tag{2.2}$$

where p is a real constant and $u_n(\tau)$ are real functions. Let us note that Eqs. (2.1), as well as Eqs. (1.1) and Eqs. (2.5) below, are invariant with respect to the Galilean transformation, i.e., if a pair $u_{1,2}(\tau)$ is a solution of Eqs. (2.1), then so is the pair

$$u_{1,2}^G(z,\tau)=u_{1,2}(\theta)\exp\left[iC\theta+i\left(p+\frac{C^2}{2}\right)z\right], \quad \theta=\tau-Cz \tag{2.3}$$

with $C=$const. We will make use of Eq. (2.3) in the next two sections.

As explained in the Introduction, Eq. (2.1) possesses symmetric [with $u_1(\tau)=u_2(\tau)$] and antisymmetric [with $u_1(\tau)=-u_2(\tau)$] solitons, which exist for $p\geq1$ and $p\geq-1$, respectively,

$$\begin{aligned} &u_1^{(s)}(\tau)=u_2^{(s)}(\tau)=\sqrt{2(p-1)}\,\mathrm{sech}[\sqrt{2(p-1)}\tau],\\ &u_2^{(\mathrm{an})}(\tau)=u_2^{(\mathrm{an})}(\tau)=\sqrt{2(p+1)}\,\mathrm{sech}[\sqrt{2(p+1)}\tau]. \end{aligned} \tag{2.4}$$

In [1] it was shown numerically that the antisymmetric and symmetric solitons become unstable for $p>-0.6$ and $p>5/3$, respectively. Thus these solitons are stable for sufficiently small values of their energy and unstable for large ones. On the other hand, the asymmetric soliton, which is created at $p=5/3$ with an already nonzero value of its energy, is unstable (with a very small instability growth rate) in a narrow region near the point of its creation, and is stable for $p>1.85$; in particular, it is stable for large energy. Let us note that an asymmetric soliton with $p\gg1$ has almost all of its energy concentrated in one of the components: $u_1(\tau)/u_2(\tau)=O(p)$ or $u_2(\tau)/u_1(\tau)=O(p)$.

The equations of pulse propagation in a single core with two orthogonal polarizations were derived in [12] (see also [13]):

$$iu_z+\tfrac{1}{2}u_{\tau\tau}+u(|u|^2+\beta|v|^2)=0,$$
$$iv_z+\tfrac{1}{2}v_{\tau\tau}+v(|v|^2+\beta|u|^2)=0. \tag{2.5}$$

One of the crucial assumptions in the derivation of these equations was that the birefringent beat length between the two eigenmodes is small compared to the nonlinear and dispersive lengths. The values of the nonlinear cross-coupling coefficient $\beta=2/3$ and $\beta=2$ correspond to linearly and circularly polarized eigenmodes of the fiber, respectively, and for a general, elliptically polarized eigenmode, $2/3<\beta<2$. Below we will refer to Eqs. (2.5) as the vector NLS equations (VNLS).

Stationary solutions of the VNLS are sought in the form

$$u(z,\tau)=u(\tau)e^{ipz},\quad v(z,\tau)=v(\tau)e^{iqz}, \tag{2.6}$$

with $u(\tau)$ and $v(\tau)$ being real. When both u and v are nonzero, the solution (2.6) is said to form a *vector soliton*, which we will denote as (u_0,v_0). Vector solitons exist in an open angle in the (p,q) plane between the straight lines [14,15]:

$$q_{\rm cr}^{\pm}=\left(\frac{\sqrt{1+8\beta}-1}{2}\right)^{\pm 2}p. \tag{2.7}$$

Outside that domain, there are only solitons of either of the following two forms:

$$(u_{00}(\tau)=\sqrt{2p}\,\operatorname{sech}\sqrt{2p}\,\tau,\quad v(\tau)=0), \tag{2.8}$$

$$(u(\tau)=0,\quad v_{00}(\tau)=\sqrt{2q}\,\operatorname{sech}\sqrt{2q}\,\tau). \tag{2.9}$$

Along the bisector $p=q$, there exists a solution with $|u|=|v|$, which can be easily found from Eqs. (2.5) and (2.6). For $q_{\rm cr}^{-}<q<q_{\rm cr}^{+}$ and $p\neq q$, the analytical form of the vector soliton is not known; however, $u_0(\tau)$ and $v_0(\tau)$ were found numerically in, e.g., [15]. Through extensive numerical simulations [8], the vector solitons of the VNLS were found to be stable for all values of p and q.

Let us now briefly summarize the relevant results of Ref. [11]. If in the DCDP one imposes the following relation between the *parallel* components in the cores:

$$u_1=\mu u_2,\quad v_1=\nu v_2,\quad \mu,\nu=\pm 1, \tag{2.10}$$

then Eqs. (1.1) reduce to equations of the form (2.5), where the control parameters p and q in Eq. (2.6) should be replaced by $(p-\mu)$ and $(q-\nu)$, respectively. Solitons of the DCDP which satisfy the reduction (2.10) were called in [11] *core-symmetric*. These core-symmetric solitons are the analogs of the symmetric and antisymmetric solitons (2.4) of the NLDC. Now, to characterize the relation between the *orthogonal* components in the same core, it is convenient to introduce a new parameter γ:

$$(q-\nu)=\gamma(p-\mu). \tag{2.11}$$

For the core-symmetric solitons, γ is the analog of the ratio q/p for the VNLS (2.5).

In [11] we also found numerically the variational approximations for the solitons of the DCDP which do not possess symmetry (2.10). Such solitons were called in [11] *core-asymmetric*. An analog of the core-asymmetric soliton is the asymmetric soliton of the NLDC.

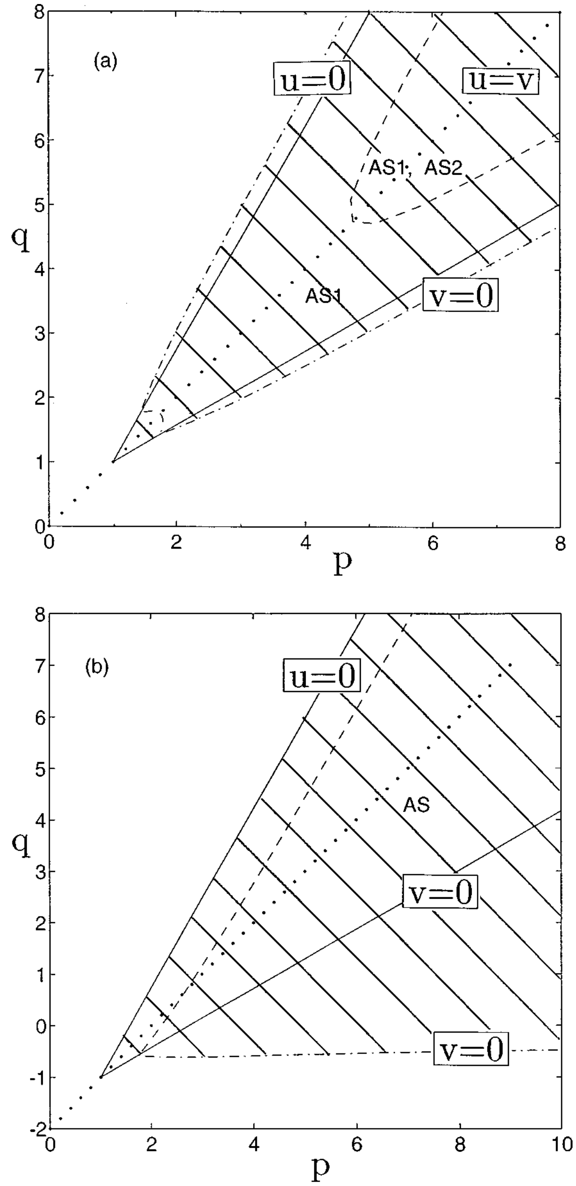

FIG. 1. Regions of existence of solutions of Eqs. (3.3) with $\beta=2/3$ in the (p,q) plane. (a) and (b) correspond to the cases $(A_1A_2>0,\ B_1B_2>0)$ and $(A_1A_2>0,\ B_1B_2<0)$, respectively; a figure for the case $(A_1A_2<0,\ B_1B_2<0)$ is not shown. Note that in (b), $A_1=A_2$ and $B_1=-B_2$ along the bisector $p=q$ only for the core-symmetric soliton.

In Figs. 1 and 2 we plotted the regions of existence of all of the various types of solitons of the DCDP, which were obtained in [11] with the variational method. Outside the shaded areas in these figures, there only exist solitons with either both u or both v components vanishing; we will not consider here such solutions because they simply reduce to the known solitons of the two-component NLDC. The core-symmetric solitons exist inside the open angles bounded by the straight lines. The dashed lines denote the *bifurcation curves*, at which the core-asymmetric solitons are created as a result of a bifurcation from the core-symmetric ones. The dash-dotted lines show where two of the components (either $u_{1,2}$ or $v_{1,2}$) of the core-asymmetric soliton vanish; thus at

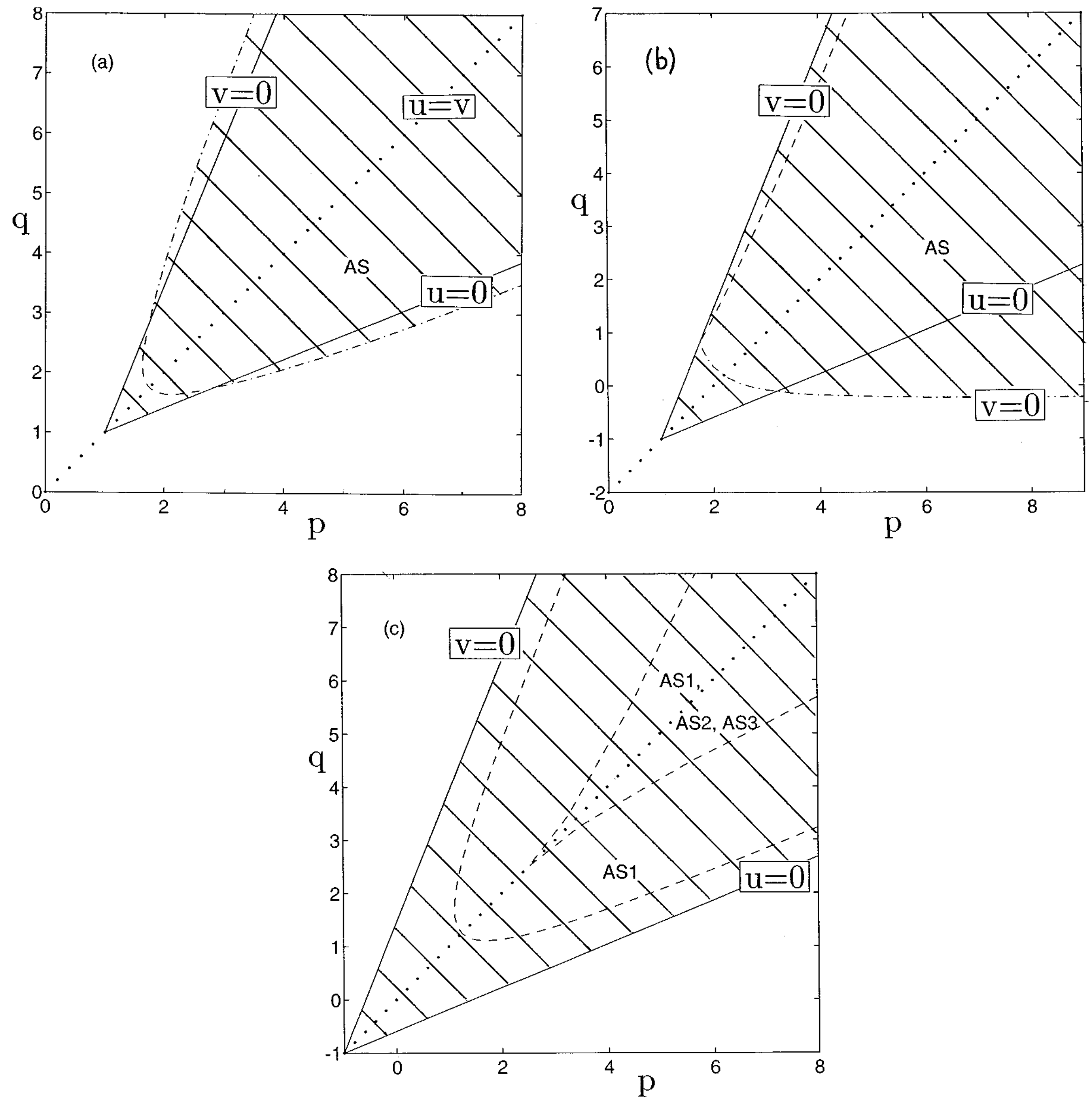

FIG. 2. Same as in Fig. 1, but $\beta=2$. Note that in (c), $A_1=-A_2$ and $B_1=-B_2$ along the bisector $p=q$ only for the core-symmetric and the core-asymmetric AS2 solitons.

these lines the solutions reduce to the asymmetric solitons of the NLDC. We will now give specific comments about each type of the core-asymmetric solitons, depicted in Figs. 1 and 2.

β=2/3;u_1u_2>0, v_1v_2>0 [Fig.1(a)]. The first type of core-asymmetric solitons exists in the open region bounded by the two dash-dotted curves and the lower dashed curve. In the next two sections we will need the asymptotic form of the solitons for large values of their energy, i.e., for $p,q\rightarrow\infty$. In this limit, one has for the first core-asymmetric soliton in Fig. 1(a)

$$(u_1,v_1)\rightarrow(u_0,v_0), \quad (u_2,v_2)\rightarrow(0,0), \tag{2.12}$$

where (u_0,v_0) is the vector soliton of the VNLS. (The asymmetric solitons always come in pairs, the two solutions in a pair differing by interchanging the subindices 1 and 2.) It is worth noting that this solution is a four-component analog of the asymmetric soliton of the NLDC, since for all sufficiently large p and q, its components satisfy the relation (cf. [11])

$$\frac{u_1}{u_2}\approx\frac{v_1}{v_2}. \tag{2.13}$$

The second core-asymmetric soliton, which exists inside the region bounded by the upper dashed curve, has the asymptotic form

$$(u_1,v_1)\rightarrow(u_0,v_0), \quad (u_2,v_2)\rightarrow(u_{00},0), \tag{2.12$'$}$$

where u_{00} and v_{00} were defined in Eqs. (2.8) and (2.9). We also remind the reader that a core-symmetric soliton with some value of γ [see Eq. (2.11)] has the same form as the vector soliton (u_0,v_0) with the ratio q/p equal to the same value γ.

$\beta=2/3; u_1u_2>0,\ v_1v_2<0$ [Fig. 1(b)]. The core-asymmetric solitons exist in the region bounded by the dashed and dash-dotted lines; their asymptotic form for $(p,q)\to\infty$ is

$$(u_1,v_1)\to(u_0,v_0),\quad (u_2,v_2)\to(0,-v_{00}). \qquad (2.14)$$

$\beta=2/3; u_1u_2<0,\ v_1v_2<0$. Only the core-symmetric solitons exist inside the open angle bounded by the straight lines (2.7), with p and q in that equation being replaced by $(p+1)$ and $(q+1)$, respectively.

$\beta=2; u_1u_2>0,\ v_1v_2>0$ [Fig. 2(a)]. The only type of core-asymmetric solitons exists in the region bounded by the dashed and dash-dotted lines; its asymptotic form is given by Eq. (2.12). The relation (2.13) is also valid in this case.

$\beta=2; u_1u_2>0,\ v_1v_2<0$ [Fig. 2(b)]. The core-asymmetric solitons which exist in the narrow strip between the dashed line and the upper solid line have a very small v component and thus are very similar to the two-component, asymmetric solitons of the NLDC. The other type of core-asymmetric solitons exists in the region bounded by the upper solid line and the dash-dotted line; its asymptotic form is

$$(u_1,v_1)\to(u_0,v_0),\quad (u_2,v_2)\to(u_{00},0). \qquad (2.15)$$

$\beta=2; u_1u_2<0,\ v_1v_2<0$ [Fig. 2(c)]. There are three different types of asymmetric solitons, which are denoted as AS1, AS2, and AS3. Their regions of existence are marked in Fig. 2(c), and their asymptotic forms are the following:

$$(u_1,v_1)\to(u_{00},0),\quad (u_2,v_2)\to(-u_0,-v_0)\quad \text{(AS1)}, \qquad (2.16a)$$

$$(u_1,v_1)\to(u_{00},0),\quad (u_2,v_2)\to(0,-v_{00})\quad \text{(AS2)}, \qquad (2.16b)$$

$$(u_1,v_1)\to(u_0,v_0),\quad (u_2,v_2)\to(0,-v_{00})\quad \text{(AS3)}. \qquad (2.16c)$$

To conclude this section, let us note that the limit $p,q\to\infty$, in which we will study the stability of the solitons of the DCDP, can only be taken formally, because, strictly speaking, it is inconsistent with the (standard) assumption of the slowly varying amplitudes, under which that model was derived. Moreover, since in this limit one also has $E\to\infty$ [see Eq. (1.3)], then the structure of the linear eigenmodes in the fiber may also change due to the strong nonlinear corrections, which will also invalidate Eqs. (1.1). However, we will now show that for realistic pulse and fiber parameters, there is a range of p and q where Eqs. (1.1) are still valid, and yet the results of our analysis are applicable. Thus we will demonstrate that taking the limit of large p and q, besides being a convenient mathematical tool, also corresponds to operating in a physically relevant range of parameters.

Following [16], we assume the following parameters for the coupler and the pulse: separation between cores $l=45\ \mu$m, diameter of a core $d=8\ \mu$m, difference between refractive indices of the core and the cladding $\Delta n=5\times10^{-3}$, carrier wavelength $\lambda\sim1\ \mu$m, pulse width $\tau_p=1$ ps. Then in Eqs. (1.1) $\kappa=0.75$ and z is normalized so as to have the coupling length $l_{\mathrm{couple}}\approx200$ m [16]. Then the requirement that the nonlinear and dispersive lengths be of the same order, i.e., $l_{\mathrm{sol}}\approx200$ m, necessitates using the pulse power $P\approx50$ W and a relatively high-dispersion fiber with $D\approx100$ ps/nm km (see, e.g., [17]). With these parameters, a solution of Eqs. (1.1) with $p=q=1$ will correspond to an approximately 1-ps-long pulse. Now, we expect that our stability results will be valid in the regions in the (p,q) plane located beyond all of the bifurcation curves (cf. Figs. 1 and 2). [Indeed, an occurrence of a pitchfork bifurcation to a solution usually indicates the appearance in the spectrum of the corresponding linearized equation of an unstable mode (see, e.g., [1]), with the instability of that mode being purely exponential rather than oscillatory. All the unstable modes that we find below are of this type, since the corresponding eigenvalues are purely imaginary.] Then, as it is seen from Figs. 1 and 2, taking $p\sim q\sim10$ is a good approximation to the asymptotic limit $p,q\gg1$. On the other hand, the pulse width scales as $1/\sqrt{p}$ and the pulse energy scales as $\sqrt{p}$ (see next section); consequently, the nonlinear and dispersive lengths both decrease as $1/p$. For $p=10$, this would yield $\tau_p\approx300$ fs, $E\approx150$ W, and $l_{\mathrm{sol}}\approx20$ m. Clearly, for such pulse widths, the approximation of the slowly varying amplitudes still holds quite well, and the pulse intensity is also sufficiently low to neglect any change in the structure of the linear eigenmodes of the fiber due to nonlinear effects.

Thus, in the following two sections, we will stay within the mathematical model given by Eqs. (1.1) and develop the stability analysis for its solitons in the limit of $p,q\to\infty$. Then, in the concluding section, we will extrapolate our results to the region of large but finite p and q.

III. STABILITY OF SOLITONS OF THE DCDP: GENERIC CASE

The idea of investigating stability of solitons of the DCDP with $p,q\gg1$ is simple. First, notice that such solitons have large amplitudes and small widths, which can be seen from the special solutions presented in Sec. II. Next, one can perform the following scaling transformation in Eqs. (1.1):

$$u=\tilde{u}/\sqrt{\varepsilon},\quad v=\tilde{v}/\sqrt{\varepsilon},\quad \tau=\tilde{\tau}\sqrt{\varepsilon},\quad z=\tilde{z}\,\varepsilon. \qquad (3.1)$$

Let so introduced amplitudes $\tilde{u}$ and $\tilde{v}$, as well as the soliton's width and dispersion length expressed in terms of the rescaled coordinates $\tilde{\tau}$ and $\tilde{z}$, respectively, have magnitudes $O(1)$. Then taking the limit $\varepsilon\ll1$ in Eq. (3.1) corresponds to the limit $p,q\gg1$ in terms of the original variables. [In fact, the propagation constants are rescaled as follows: $p=\tilde{p}/\varepsilon$ and $q=\tilde{q}/\varepsilon$, where $\tilde{p},\tilde{q}=O(1)$.] On the other hand, the tilded quantities satisfy Eqs. (1.1) with $\tilde{\kappa}=\varepsilon\kappa$. Since using the above scaling transformation, one can always rescale a nonzero κ in Eqs. (1.1) to unity, we will write in what follows $\tilde{\kappa}=\varepsilon$ without restricting the generality. Below we will also omit the tilde sign. Thus we have shown that the limit $p,q\gg1$, κ fixed in Eqs. (1.1) is equivalent to the limit p,q fixed, $\kappa\ll1$, which is the limit of small coupling between the cores. Then the problem of stability of solitons of the DCDP in the former limit is reduced to the problem of stability of solitons in a single, strongly birefringent fiber, with the perturbation being the linear coupling to the other fiber.

If one formally sets $\kappa\equiv\varepsilon=0$ in Eqs. (1.1), then the resulting equations will describe two *uncoupled* cores, with two orthogonal polarizations in each. The equation for each core is the VNLS (2.5). A vector soliton of the VNLS is invariant with respect to the transformation

$$u(\tau)\rightarrow u(\tau-\tau_0)e^{i\varphi_u}, \quad v(\tau)\rightarrow v(\tau-\tau_0)e^{i\varphi_v}, \tag{3.2}$$

where τ_0, φ_u, and φ_v are arbitrary constants. Therefore the vector soliton (2.6) has three Goldstone modes which correspond to the infinitesimal shifts of τ_0, φ_u, and φ_v. Altogether, there are six Goldstone modes for two independent vector solitons in the two uncoupled cores. Such modes are always neutrally stable. Now, when one couples the two cores by allowing the linear coupling constant ε to be nonzero, there remain only three Goldstone modes (one corresponds to the shift of the common center of all the four components, and the other two to the shifts of the phases of the $u_{1,2}$ and $v_{1,2}$ components). The other three "formerly Goldstone" modes need no longer be neutrally stable, and, in general, they will become modes of the discrete spectrum with nonzero eigenvalues. Then for $\varepsilon\ll 1$, one can find these eigenvalues by means of a perturbation theory, and thus establish the stability or instability of the soliton. We will now give the details of these calculations.

Let

$$(u_{10}(\tau)e^{ipz},v_{10}(\tau)e^{iqz},u_{20}(\tau)e^{ipz},v_{20}(\tau)e^{iqz})^T, \tag{3.3}$$

where the superscript T indicates the matrix transpose, be a four-component vector soliton of Eqs. (1.1) with $\varepsilon=0$. Note that even though for $\varepsilon=0$, the vector soliton in the first core is not coupled to the vector soliton in the other core, we have required that the propagation constants of the parallel components of both solitons be equal, because this is the solution of interest for $\varepsilon\rightarrow+0$. When $0<\varepsilon\ll 1$, then we can expand the profile of the stationary soliton by

$$u_n(\tau)=u_{n0}(\tau)+\varepsilon u_{n1}(\tau)+\cdots,$$

$$v_n(\tau)=v_{n0}(\tau)+\varepsilon v_{n1}(\tau)+\cdots, \quad n=1,2\ . \tag{3.4}$$

Let us introduce the eight-component vectors

$$\mathbf{w}_m(\tau)=(u_{1m},u_{1m}^*,v_{1m},v_{1m}^*,u_{2m},u_{2m}^*,v_{2m},v_{2m}^*)^T,$$

$$m=0,1\ .$$

[In fact, $u_{10}(\tau)$, $u_{11}(\tau)$, etc., are real. We only introduce their complex conjugates here for notational convenience later on.] In order to make the ensuing formulas more compact, we will use the notations

$$(I\vec{x}_1,I\vec{x}_2,I\vec{x}_3,I\vec{x}_4)^T \quad \text{for} \quad (x_1,x_1,x_2,x_2,x_3,x_3,x_4,x_4)^T$$

and

$$(\sigma_3\vec{x}_1,\sigma_3\vec{x}_2,\sigma_3\vec{x}_3,\sigma_3\vec{x}_4)^T$$

$$\text{for} \ (x_1,-x_1,x_2,-x_2,x_3,-x_3,x_4,-x_4)^T.$$

Above, I is the 2×2 identity matrix, $\vec{x}_j=(x_j,x_j)^T$, and the Pauli matrices are

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Thus, for instance, if all of the components of the vector $\mathbf{w}_m$ $(m=0,1)$ are real, then

$$\mathbf{w}_m=(I\vec{u}_{1m},I\vec{v}_{1m},I\vec{u}_{2m},I\vec{v}_{2m})^T, \quad m=0,1\ .$$

Also, we will use the notation $\hat{\sigma}_3$ for a block-diagonal matrix $\mathrm{diag}(\sigma_3,\sigma_3,\sigma_3,\sigma_3)$.

Substituting the expansion (3.4) into Eqs. (1.1) with $\kappa=\varepsilon$, one finds that vector $\mathbf{w}_1$ satisfies the following equation:

$$\L_0\mathbf{w}_1\equiv\begin{pmatrix}L_1 & 0\\ 0 & L_2\end{pmatrix}\mathbf{w}_1=R[\mathbf{w}_0], \tag{3.5}$$

where

$$L_n=\begin{pmatrix}\left(\frac{1}{2}\partial_\tau^2-p\right)\sigma_3 & 0\\ 0 & \left(\frac{1}{2}\partial_\tau^2-q\right)\sigma_3\end{pmatrix}+\begin{pmatrix}(2u_{n0}^2+\beta v_{n0}^2)\sigma_3+u_{n0}^2 i\sigma_2 & \beta u_{n0}v_{n0}(\sigma_3+i\sigma_2)\\ \beta u_{n0}v_{n0}(\sigma_3+i\sigma_2) & (2v_{n0}^2+\beta u_{n0}^2)\sigma_3+v_{n0}^2 i\sigma_2\end{pmatrix}, \quad n=1,2 \tag{3.6a}$$

and

$$R[(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3,\mathbf{y}_4)^T]\equiv-(\mathbf{y}_3\sigma_3,\mathbf{y}_4\sigma_3,\mathbf{y}_1\sigma_3,\mathbf{y}_2\sigma_3)^T. \tag{3.6b}$$

In Eq. (3.6b), $\mathbf{y}_1$, etc. are arbitrary two-component row vectors. Note that the operator $\L_0$ in the left hand side of Eq. (3.5) is the linearized operator of Eqs. (1.1) on the background of the soliton (3.3), which is a solution of Eqs. (1.1) with $\kappa=0$. The operator R in the right hand side of Eq. (3.5) arises from the terms describing the small linear coupling between the two cores. As will be shown below [after Eq. (3.20)], the right hand side of Eq. (3.5) is orthogonal to the zero modes of the operator $\L_0$, so that the solvability condition for Eq. (3.5) is always satisfied. Then, the stationary corrections u_{n1},v_{n1}, $n=1,2$ to the soliton will be exponentially decaying functions of $|\tau|$. Equation (3.5) is, in general, quite complex; however, we will *not* need its solution in explicit form.

When none of the components of soliton (3.3) are zero, then for $\varepsilon=0$, that soliton has six Goldstone modes of the following form:

$$\phi_0^{(1,g)}=\begin{pmatrix} I\vec{u}_{10,\,\tau} \\ I\vec{v}_{10,\,\tau} \\ gI\vec{u}_{20,\,\tau} \\ gI\vec{v}_{20,\,\tau} \end{pmatrix}, \quad \phi_0^{(2,g)}=\begin{pmatrix} \sigma_3 i\vec{u}_{10} \\ \vec{0} \\ g\sigma_3 i\vec{u}_{20} \\ \vec{0} \end{pmatrix},$$

$$\phi_0^{(3,g)}=\begin{pmatrix} \vec{0} \\ \sigma_3 i\vec{v}_{10} \\ \vec{0} \\ g\sigma_3 i\vec{v}_{20} \end{pmatrix}, \tag{3.7}$$

where $g=\pm 1$. The modes with $g=+1$ correspond to the equal shifts of the parameters τ_0, φ_u, and φ_v, defined in Eq. (3.2), for the uncoupled solitons in the two cores, whereas the modes with $g=-1$ correspond to the equal in magnitude but oppositely directed such shifts. All these Goldstone modes satisfy the equation

$$\L_0\phi_0^{(j,g)}=0, \quad j=1,2,3\,. \tag{3.8}$$

Thus the eigenvalue λ of the operator $\L_0$ will have multiplicity of *12* for $\lambda=0$, since for each of the above six Goldstone modes, the spectrum of the operator $\L_0$ is doubly degenerate. Indeed, to each of these six modes, there corresponds a so-called associate mode, $\phi_{\mathcal{D}}^{(j,g)}$ (also called a derivative state, cf. [18–20]), which satisfies

$$\alpha_j \L_0\phi_{\mathcal{D}}^{(j,g)}=\phi_0^{(j,g)}, \tag{3.9a}$$

where the normalization constants $\alpha_{1,2,3}$ have been chosen to be

$$\alpha_1=-i, \quad \alpha_2=\alpha_3=i. \tag{3.9b}$$

The explicit form of the associate modes is the following (cf., e.g., [20]):

$$\phi_{\mathcal{D}}^{(1,g)}=\begin{pmatrix} \sigma_3\vec{u}_{10,\,C} \\ \sigma_3\vec{v}_{10,\,C} \\ g\sigma_3\vec{u}_{20,\,C} \\ g\sigma_3\vec{v}_{20,\,C} \end{pmatrix}, \quad \phi_{\mathcal{D}}^{(2,g)}=\begin{pmatrix} I\vec{u}_{10,\,p} \\ I\vec{v}_{10,\,p} \\ gI\vec{u}_{20,\,p} \\ gI\vec{v}_{20,\,p} \end{pmatrix},$$

$$\phi_{\mathcal{D}}^{(3,g)}=\begin{pmatrix} I\vec{u}_{10,\,q} \\ I\vec{v}_{10,\,q} \\ gI\vec{u}_{20,\,q} \\ gI\vec{v}_{20,\,q} \end{pmatrix}, \tag{3.10}$$

where

$$u_{10,\,C}=\frac{\partial u_{10}^G}{\partial C}\bigg|_{\substack{\theta=\text{const}\\ C=0}}, \quad u_{10,\,p}=\frac{\partial u_{10}}{\partial p}\bigg|_{q=\text{const}},$$

$$u_{10,\,q}=\frac{\partial u_{10}}{\partial q}\bigg|_{p=\text{const}}, \text{ etc.}$$

Here the soliton u_{10}^G is obtained from u_{10} by the Galilean transformation (2.3), so that C is the soliton velocity. According to this definition, it is easy to obtain that

$$u_{10,\,C}=i\frac{\tau}{2}u_{10}, \quad \text{etc.} \tag{3.11}$$

However, no such relation can be given for $u_{10,p}$, $u_{10,q}$, etc., since the explicit dependence of the components of the vector soliton (3.3) on p and q is not known.

Now, for $0<\varepsilon\ll 1$, the three modes $\phi_0^{(j,+1)}$ (i.e., for $g=+1$) continue to remain as Goldstone modes. The other three (we remind the reader that we consider the case when none of the components of the soliton (3.3) are zero; the case when one or more of those components vanishes will be commented on later) modes, with $g=-1$, may acquire nonzero eigenvalues as modes of the discrete spectrum of the linearized Eqs. (1.1). Since each $\phi_0^{(j,-1)}$ is doubly degenerate, as explained above, then for $\varepsilon\neq 0$, each must, in general, split and give rise to two modes of the discrete spectrum.

To determine how the eigenvalues of these new modes shift, one needs to linearize Eqs. (1.1) on the background of the soliton (3.4). Thus we let

$$u_{1,2}(\tau,z)=[u_{1,2}(\tau)+U_{1,2}(\tau,z)]e^{ipz},$$

$$v_{1,2}(\tau,z)=[v_{1,2}(\tau)+V_{1,2}(\tau,z)]e^{iqz}, \tag{3.12}$$

where $|U_{1,2}|\ll|u_{1,2}|$ and $|V_{1,2}|\ll|v_{1,2}|$, and the magnitudes of $U_{1,2}$ and $V_{1,2}$ are *not* related to the value of ε. Note that we have performed *two* expansions. The first expansion, given by Eqs. (3.4) and (3.5), defined how the stationary form of the soliton shifted for $\varepsilon\neq 0$. In the second expansion, Eq. (3.12), we consider *arbitrary* (but small) perturbations of the profile of the soliton (3.4).

For the eight-component vector

$$\mathbf{W}_1(\tau,z)\equiv(U_1,U_1^*,V_1,V_1^*,U_2,U_2^*,V_2,V_2^*)^T$$

we obtain the following equation:

$$(i\partial_z+\L_0)\mathbf{W}_1=\varepsilon\{R[\mathbf{W}_1]-(\Delta\L_0)\mathbf{W}_1\}, \tag{3.13}$$

where we have dropped terms of order $O(\varepsilon^2)$ and higher. In the last equation, the notation $(\Delta\L_0)$ is the following:

$$(\Delta\L_0)\equiv\begin{pmatrix} \Delta L_1 & 0 \\ 0 & \Delta L_2 \end{pmatrix}=\L_0[(u_1,v_1,u_2,v_2)]$$
$$-\L_0[(u_{10},v_{10},u_{20},v_{20})], \tag{3.14a}$$

where the notation $\L_0[(a,b,c,d)]$ means that a, b, c, and d are substituted for u_{10}, v_{10}, u_{20}, and v_{20}, respectively, in Eq. (3.6a). One obtains

$$\Delta L_n = \begin{pmatrix} 2(2u_{n0}u_{n1}+\beta v_{n0}v_{n1})\sigma_3+2u_{n0}u_{n1}i\sigma_2 & \beta(u_{n0}v_{n1}+u_{n1}v_{n0})(\sigma_3+i\sigma_2) \\ \beta(u_{n0}v_{n1}+u_{n1}v_{n0})(\sigma_3+i\sigma_2) & 2(2v_{n0}v_{n1}+\beta u_{n0}u_{n1})\sigma_3+2v_{n0}v_{n1}i\sigma_2 \end{pmatrix}, \tag{3.14b}$$

where $n=1,\ 2$, and u_{11}, etc. can be found from Eq. (3.5). It is seen from Eq. (3.13) that the introduction of a small linear coupling between two, originally uncoupled, cores can affect the stability of the solitons in such a system in two different ways. First, the coupling terms alter the form of the corresponding linearized equations, which is accounted for by the first term in the right hand side of Eq. (3.13). Second, the stationary solitons in the presence of a small coupling are different from those with zero coupling, cf. Eqs. (3.5) and (3.4). Therefore linearization now needs to be performed on the background of a soliton that is slightly different from that in the case of zero coupling. This leads to the occurrence of the second term in the right hand side of Eq. (3.13).

To solve Eq. (3.13), we use the separation of variables:

$$\mathbf{W}_1(\tau,z)=\phi(\tau)e^{i\lambda z},$$

and obtain:

$$Ł_0\phi=\lambda\phi+\varepsilon\{R[\phi]-(\Delta Ł_0)\phi\}. \tag{3.15}$$

Thus, if there exists at least one mode $\phi_0^{(j,-1)}$ which has an eigenvalue with $\mathrm{Im}\lambda<0$ for $0<\varepsilon\ll 1$, then the corresponding soliton (3.4) is linearly unstable.

To solve Eq. (3.15) by successive approximations for $0<\varepsilon\ll 1$, we assume the following expansions:

$$\phi^{(j,g)}=\phi_0^{(j,g)}+\sqrt{\varepsilon}\,\phi_1^{(j,g)}+\varepsilon\,\phi_2^{(j,g)}+\cdots,$$

$$\lambda^{(j,g)}=\sqrt{\varepsilon}\lambda_1^{(j,g)}+\varepsilon\lambda_2^{(j,g)}+\cdots. \tag{3.16}$$

The expansion in powers of $\sqrt{\varepsilon}$ rather than ε is being performed because each of the Goldstone modes (3.7) is doubly degenerate. Substituting Eq. (3.16) into Eq. (3.15) and using Eq. (3.8), one obtains in the first order:

$$Ł_0\phi_1^{(j,g)}=\lambda_1^{(j,g)}\phi_0^{(j,g)}. \tag{3.17}$$

From Eq. (3.9), a particular solution of the last equation is

$$\phi_1^{(j,g)}=\lambda_1^{(j,g)}\alpha_j\phi_{\mathcal{D}}^{(j,g)}. \tag{3.18}$$

In the next order, one obtains from Eqs. (3.15), (3.16), and (3.18)

$$Ł_0\phi_2^{(j,g)}=(\lambda_1^{(j,g)})^2\alpha_j\phi_{\mathcal{D}}^{(j,g)}+\lambda_2^{(j,g)}\phi_0^{(j,g)}+\{R[\phi_0^{(j,g)}]-(\Delta Ł_0)\phi_0^{(j,g)}\}. \tag{3.19}$$

Now multiply Eq. (3.19) by $(\phi_0^{(j,g)})^T\hat{\sigma}_3$ on the left, integrate over τ, and use the fact that $\hat{\sigma}_3 Ł_0$ is a Hermitian operator. This gives the following solvability condition for Eq. (3.19):

$$(\lambda_1^{(j,g)})^2\alpha_j\langle\phi_0^{(j,g)}|\hat{\sigma}_3|\phi_{\mathcal{D}}^{(j,g)}\rangle=\langle\phi_0^{(j,g)}|\hat{\sigma}_3(\Delta Ł_0)|\phi_0^{(j,g)}\rangle-\langle\phi_0^{(j,g)}|\hat{\sigma}_3|R[\phi_0^{(j,g)}]\rangle, \tag{3.20}$$

where for any eight-component vector functions $f(\tau)$ and $h(\tau)$,

$$\langle f|\hat{\sigma}_3|h\rangle\equiv\int_{-\infty}^{\infty}f^T(\tau)\hat{\sigma}_3 h(\tau)d\tau.$$

Similarly, one establishes that the solvability condition for Eq. (3.5), which determines the stationary first-order corrections $u_{n1},v_{n1},\ n=1,2$ [see Eq. (3.4)], always holds.

Now note that the term in angle brackets in the left hand side of Eq. (3.20) is independent of g, and so is the first term in the right hand side due to the block-diagonal structure of $(\Delta Ł_0)$ [see Eq. (3.14a)], while the second term in the right hand side is proportional to g, see Eqs. (3.7) and (3.6b). Furthermore, we know that the modes with $g=+1$ remain the Goldstone modes for $\varepsilon\neq 0$, and therefore their eigenvalues must not shift from zero, for all orders of ε. In particular, $\lambda_1^{(j,+1)}=0$ for $j=1,2,3$. Then from Eq. (3.20) it follows that the second term in the right hand side is exactly cancelled by the first one when $g=+1$, and so for $g=-1$, the right hand side must be twice the second term. Thus Eq. (3.20) for $g=-1$ can be rewritten as follows:

$$(\lambda_1^{(j,-1)})^2\alpha_j\langle\phi_0^{(j,-1)}|\hat{\sigma}_3|\phi_{\mathcal{D}}^{(j,-1)}\rangle=-2\,\langle\phi_0^{(j,-1)}|\hat{\sigma}_3|R[\phi_0^{(j,-1)}]\rangle. \tag{3.21}$$

Substituting Eqs. (3.9b), (3.10), and (3.11) into Eqs. (3.21), one obtains equations which, in the first approximation, determine the eigenvalues of the modes of the discrete spectrum:

$$(\lambda_1^{(1,-1)})^2(E_u+E_v)=16\int_{-\infty}^{\infty}(u_{10,\,\tau}u_{20,\,\tau}+v_{10,\,\tau}v_{20,\,\tau})d\tau, \tag{3.22a}$$

$$(\lambda_1^{(2,-1)})^2\frac{\partial E_u}{\partial p}\bigg|_{q=\mathrm{const}}=-8\int_{-\infty}^{\infty}u_{10}u_{20}d\tau, \tag{3.22b}$$

$$(\lambda_1^{(3,-1)})^2\frac{\partial E_v}{\partial q}\bigg|_{p=\mathrm{const}}=-8\int_{-\infty}^{\infty}v_{10}v_{20}d\tau, \tag{3.22c}$$

where E_u and E_v were defined in Eq. (1.3). We will present the conclusions regarding the stability of the solitons, which follow from Eqs. (3.22), in Sec. V.

Let us remark again that Eqs. (3.22) are valid only when $u_{10}u_{20}\neq 0$ and $v_{10}v_{20}\neq 0$, since otherwise the total number of Goldstone modes will be less than six. Indeed, if, for example, $u_{10}u_{20}=0$ and $v_{10}v_{20}\neq 0$, then $\phi_0^{(2,-1)}\equiv\phi_0^{(2,+1)}$, and also $\phi_{\mathcal{D}}^{(2,-1)}\equiv\phi_{\mathcal{D}}^{(2,+1)}$ [to see that the latter relation is true, one needs to notice that when $u_0\equiv 0$, then $v_{0,p}\equiv 0$ in Eqs. (2.5)]. Thus in this case one has five Goldstone modes instead of six in Eq. (3.7). However, we can still use Eqs. (3.22) in this case (as well as in the case when $u_{10}u_{20}\neq 0$ and $v_{10}v_{20}=0$) to determine stability, as long as at least one of the eigenvalues has an imaginary part, in which case the corresponding soliton is unstable.

Now, when $u_{10}u_{20}=0$ and $v_{10}v_{20}=0$, Eqs. (3.22) do not yield information regarding the stability of the solitons. In this case, one needs to consider two different possibilities. First, when $u_{20}=v_{20}=0$ and $u_{10}\neq 0\neq v_{10}$, Eqs. (3.7) yield only three Goldstone modes. Since for $0<\varepsilon\ll 1$, these modes continue to be Goldstone modes, then no modes with nonzero eigenvalues exist in the discrete spectrum, and so the background soliton is stable. This situation is realized for the first core-asymmetric soliton in Fig. 1(a) and the core-asymmetric soliton in Fig. 2(a) [see Eq. (2.12)]. Thus those solitons, which are the four-component analogs of the asymmetric soliton of the NLDC (2.1), are stable for $p,q\gg 1$. The second possibility is when $u_{20}=v_{10}=0$ and $u_{10}\neq 0\neq v_{20}$, which is realized for the AS2 soliton in Fig. 2(c). In this case, the mode $\phi_0^{(1,-1)}$, which corresponds to the oppositely directed shifts of the centers of the solitons in the two uncoupled cores, is different from the mode $\phi_0^{(1,+1)}$, corresponding to the similarly directed shifts. Therefore Eqs. (3.7) yield four Goldstone modes, of which one, $\phi_0^{(1,-1)}$, may acquire a nonzero eigenvalue for $\varepsilon\neq 0$. To find that eigenvalue, one needs to use an expansion different from Eqs. (3.16). We do this in the next section.

IV. PROOF THAT THE AS2 SOLITON IN FIG. 2(c) IS UNSTABLE

As was explained at the end of the preceding section, in the case of the AS2 soliton in Fig. 2(c), the only ''formerly Goldstone'' mode which may acquire a nonzero eigenvalue for $\varepsilon\neq 0$ is $\phi_0^{(1,-1)}$ in Eq. (3.7). From Eqs. (3.7), (3.10), and (2.16b) one finds its explicit form, as well as that of the associated mode:

$$\phi_0^{(1,g)}=\begin{pmatrix} I\vec{u}_{00,\tau}\\ \vec{0}\\ \vec{0}\\ gI\vec{v}_{00,\tau}\end{pmatrix},\qquad \phi_{\mathcal{D}}^{(1,g)}=\begin{pmatrix} \sigma_3\vec{u}_{00,C}\\ \vec{0}\\ \vec{0}\\ g\sigma_3\vec{v}_{00,C}\end{pmatrix},\tag{4.1}$$

with $g=-1$. The mode with $g=+1$ continues to be a Goldstone mode even for $\varepsilon\neq 0$. We will not be referring to the modes $\phi_0^{(2,g)}$ and $\phi_0^{(3,g)}$ in this section, and therefore below we will refer to the modes $\phi_0^{(1,g)}$ simply as to $\phi_0^{(g)}$. Note that the relative sign of the nonzero u and v components of the soliton is unimportant, and so we chose both these components to be of the same sign.

Instead of Eq. (3.16), one should now use another expansion for the eigenmode and its eigenvalue:

$$\phi^{(g)}=\phi_0^{(g)}+\varepsilon\phi_1^{(g)}+\varepsilon^2\phi_2^{(g)}+\cdots,$$

$$\lambda^{(g)}=\varepsilon\lambda_1^{(g)}+\varepsilon^2\lambda_2^{(g)}+\cdots .\tag{4.2}$$

One also has to take into account the second-order correction terms to the stationary solution:

$$u_n(\tau)=u_{n0}(\tau)+\varepsilon u_{n1}(\tau)+\varepsilon^2u_{n2}(\tau)+\cdots,$$
$$v_n(\tau)=v_{n0}(\tau)+\varepsilon v_{n1}(\tau)+\varepsilon^2v_{n2}(\tau)+\cdots ,\tag{4.3}$$

for $n=1,2$; recall that the above solutions are real. From Eqs. (3.5), (3.6), and (2.16b) one finds that

$$[(\tfrac{1}{2}\partial_\tau^2-q)+\beta u_{00}^2]v_{11}=-v_{00},$$
$$[(\tfrac{1}{2}\partial_\tau^2-p)+\beta v_{00}^2]u_{21}=-u_{00},\tag{4.4a}$$

$$u_{11}=v_{21}=0 .\tag{4.4b}$$

One can also show that the second-order correction terms, which satisfy an equation similar to Eq. (3.5), satisfy the relations

$$u_{12}\neq 0\neq v_{22},\quad v_{12}=0=u_{22},\tag{4.4c}$$

but we will not need the explicit form of those corrections.

Now, up to the order $O(\varepsilon^2)$, the equation for the eigenmode is

$$\text{Ł}_0\phi=\lambda\phi+\varepsilon\{R[\phi]-(\Delta_1\text{Ł}_0)\phi\}-\varepsilon^2(\Delta_2\text{Ł}_0)\phi,\tag{4.5}$$

where $(\Delta_1\text{Ł}_0)$ is the same quantity which was defined in Eq. (3.14), and $(\Delta_2\text{Ł}_0)$ is the second-order correction to the operator Ł_0, which is due both to the second-order terms of the expansion (4.3) and the quadratic combinations of the first-order terms of that expansion. Both $(\Delta_1\text{Ł}_0)$ and $(\Delta_2\text{Ł}_0)$ have the block-diagonal structure shown in Eq. (3.14a); however, within the nonzero 4×4 blocks, the structure of these matrices is different, namely,

$$\Delta_1L_1=\begin{pmatrix}0 & \beta u_{00}v_{11}(\sigma_3+i\sigma_2)\\ \beta u_{00}v_{11}(\sigma_3+i\sigma_2) & 0\end{pmatrix},$$
$$\Delta_1L_2=\begin{pmatrix}0 & \beta v_{00}u_{21}(\sigma_3+i\sigma_2)\\ \beta v_{00}u_{21}(\sigma_3+i\sigma_2) & 0\end{pmatrix},\tag{4.6a}$$

and

$$\Delta_2L_n=\begin{pmatrix}* & 0\\ 0 & *\end{pmatrix},\quad n=1,2\tag{4.6b}$$

where the asterisk stands for a nonzero 2×2 block, whose explicit form we will not need. Expressions in Eq. (4.6) were obtained with the use of Eqs. (3.6a), (2.16b), and (4.4).

Proceeding with the solution of Eq. (4.5), one obtains in the first order

$$\text{Ł}_0\phi_1^{(g)}=\lambda_1^{(g)}\phi_0^{(g)}+R[\phi_0^{(g)}]-(\Delta_1\text{Ł}_0)\phi_0^{(g)}.\tag{4.7}$$

Solvability condition for this equation is satisfied. Indeed, if it were not, then the expansion for the eigenvalue would have been given by Eq. (3.16) rather than Eq. (4.2), and the latter is not the case, since then one of Eqs. (3.22) would have provided conclusive information about the stability of the AS2 soliton in the first order. Now, a particular solution to Eq. (4.7) has the form

$$\phi_1^{(g)} = \alpha\lambda_1^{(g)}\phi_{\mathcal{D}}^{(g)} + \phi_R^{(g)} - \phi_\Delta^{(g)}, \tag{4.8}$$

where $\alpha \equiv \alpha_1 = -i$ [see Eq. (3.9b)] and

$$\L_0\phi_R^{(g)} = R[\phi_0^{(g)}], \quad \L_0\phi_\Delta^{(g)} = (\Delta_1\L_0)\phi_0^{(g)}. \tag{4.9}$$

Using Eqs. (3.5), (2.16b), (4.1), and (4.6a), one finds that

$$\phi_R^{(g)} = -\begin{pmatrix} \vec{0} \\ gI\vec{\psi}_1 \\ I\vec{\psi}_2 \\ \vec{0} \end{pmatrix}, \quad \phi_\Delta^{(g)} = \begin{pmatrix} \vec{0} \\ I\vec{\psi}_3 \\ gI\vec{\psi}_4 \\ \vec{0} \end{pmatrix}, \tag{4.10}$$

where ψ_n, $n=1,\ldots,4$, are solutions of the following equations:

$$[(\tfrac{1}{2}\partial_\tau^2 - q) + \beta u_{00}^2]\psi_1 = v_{00,\tau}, \tag{4.11a}$$

$$[(\tfrac{1}{2}\partial_\tau^2 - p) + \beta v_{00}^2]\psi_2 = u_{00,\tau}, \tag{4.11b}$$

$$[(\tfrac{1}{2}\partial_\tau^2 - q) + \beta u_{00}^2]\psi_3 = \beta(u_{00}^2)_\tau v_{11}, \tag{4.11c}$$

$$[(\tfrac{1}{2}\partial_\tau^2 - p) + \beta v_{00}^2]\psi_4 = \beta(v_{00}^2)_\tau u_{21}. \tag{4.11d}$$

In the next order, one finds from Eq. (4.5)

$$\L_0\phi_2^{(g)} = \lambda_2^{(g)}\phi_0^{(g)} + \lambda_1^{(g)}\phi_1^{(g)} + R[\phi_1^{(g)}] - (\Delta_1\L_0)\phi_1^{(g)} - (\Delta_2\L_0)\phi_0^{(g)}. \tag{4.12}$$

Using Eqs. (4.1), (4.8), (4.10), (3.6b), and (4.6), one can show that only the following terms will contribute to the solvability condition of Eq. (4.12):

$$(\lambda_1^{(g)})^2\alpha\langle\phi_0^{(g)}|\hat{\sigma}_3|\phi_{\mathcal{D}}^{(g)}\rangle = \langle\phi_0^{(g)}|\hat{\sigma}_3|\{R[\phi_\Delta^{(g)}] + (\Delta_1\L_0)\phi_R^{(g)}\}\rangle - \langle\phi_0^{(g)}|\hat{\sigma}_3|\{R[\phi_R^{(g)}] + (\Delta_1\L_0)\phi_\Delta^{(g)} - (\Delta_2\L_0)\phi_0^{(g)}\}\rangle. \tag{4.13}$$

One can verify, using Eqs. (4.1), (4.10), (3.6b), and (3.14), that the first term in the right hand side of Eq. (4.13) is proportional to g, while the second one is independent of it. Since for $g=+1$, the two terms must exactly cancel each other (cf. Sec. III), then for $g=-1$, the right hand side is twice the first term. Then, using Eqs. (4.10) and (4.6a), and also (4.1) and (3.11), one obtains

$$\begin{aligned}(\lambda_1^{(-1)})^2(E_u+E_v) &= -4\int_{-\infty}^{\infty}\{[\psi_4 u_{00,\tau} + \beta(v_{00}^2)_\tau u_{21}\psi_2] \\ &\quad + [\psi_3 v_{00,\tau} + \beta(u_{00}^2)_\tau v_{11}\psi_1]\}d\tau \\ &= -8\int_{-\infty}^{\infty}[\beta(v_{00}^2)_\tau u_{21}\psi_2 \\ &\quad + \beta(u_{00}^2)_\tau v_{11}\psi_1]d\tau, \end{aligned} \tag{4.14}$$

where in deriving the last line, we have used Eqs. (4.4a) and (4.11).

We numerically solved Eqs. (4.4a), (4.11a), and (4.11b) for various values of the ratio (q/p) in the interval $(\gamma_{\rm cr}^-, \gamma_{\rm cr}^+)$ [cf. Eqs. (2.11), (2.7)] and found that the right hand side of Eq. (4.14) was always negative. This means that $(\lambda_1^{(-1)})^2<0$, and so the AS2 soliton in Fig. 2(c) is unstable in the limit of large p and q.

V. DISCUSSION OF THE RESULTS AND CONCLUSIONS

In this section, we will first use Eqs. (3.22) to derive the stability properties of the solitons of the DCDP. Then we will formulate open problems regarding the stability of solitons of the DCDP.

$\beta=2/3, u_1u_2>0, v_1v_2>0$. As it was shown at the end of Sec. III, the first core-asymmetric soliton in Fig. 1(a) is stable for $p,q \gg 1$. In analogy with the results for Eqs. (2.1), we surmise that that soliton is stable either immediately or shortly after the first bifurcation curve in Fig. 1(a). The core-symmetric soliton and the second core-asymmetric soliton [see Eq. (2.12′)] are unstable since for them $(\lambda_1^{(2,-1)})^2<0$ in Eqs. (3.22) [for the core-symmetric soliton, $(\lambda_1^{(3,-1)})^2<0$ as well]. In arriving at this conclusion, we have used the fact that for all the types of solitons in the case considered, $\partial E_u/\partial p|_{q=\rm const}>0$, which can be seen from Fig. 1(a) and Eq. (2.7) with $\beta<1$. We note that the instability of the core-symmetric soliton in this case is analogous to that of the symmetric solution of the NLDC, and so we surmise that for finite p and q, the core-symmetric soliton is stable before the first bifurcation curve and unstable beyond it. Then the second core-asymmetric soliton must be unstable for all p and q since it comes into existence as a result of a pitchfork bifurcation from the already unstable core-symmetric soliton.

$\beta=2/3, u_1u_2>0, v_1v_2<0$. For the core-symmetric soliton, $(\lambda_1^{(2,-1)})^2<0$, and also for the core-asymmetric soliton, $(\lambda_1^{(1,-1)})^2<0$ [see Eq. (2.14)]. Thus both types of solitons in this case are linearly unstable for large p and q. However, in analogy with the results for the NLDC, the core-symmetric soliton with sufficiently low energy may be stable.

$\beta=2/3, u_1u_2<0, v_1v_2<0$. The core-symmetric soliton in this case has $(\lambda_1^{(1,-1)})^2<0$, and thus it is unstable. This is analogous to the instability of the antisymmetric soliton of the NLDC.

$\beta=2$ $u_1u_2>0$, $v_1v_2>0$. As was shown at the end of Sec. III, the core-asymmetric soliton is stable, similar to the

case when $\beta=2/3$. Now, unlike that latter case, here one has for the core-symmetric soliton

$$\left.\frac{\partial E_u}{\partial p}\right|_{q=\mathrm{const}}<0,\quad \left.\frac{\partial E_v}{\partial q}\right|_{p=\mathrm{const}}<0\,. \tag{5.1}$$

This can be seen from Fig. 2(a) and Eq. (2.7) with $\beta>1$. Therefore all the eigenvalues for the core-symmetric soliton are real in the first approximation. However, that solution still has a chance to be unstable, if it turns out that in the next approximation, at least one of the $\lambda_2^{(j,-1)}$'s is imaginary. However, in that case the rate of instability, if one returns to the initial evolution variable z [see Eq. (3.1)], will have the order of magnitude $O(1)$, while in all the cases considered so far it had the order of magnitude $O(1/\sqrt{\varepsilon})$ [cf. Eq. (3.16)].

$\beta=2$, $u_1u_2>0$, $v_1v_2<0$. For the core-symmetric soliton, Eqs. (5.1) hold, and therefore $(\lambda_1^{(3,-1)})^2<0$ for this type of soliton. For the core-asymmetric soliton which exists on the right hand side of the dash-dotted curve in Fig. 2(b), we found numerically that $\partial E_u/\partial p|_{q=\mathrm{const}}>0$, and so for this type of soliton, $(\lambda_1^{(2,-1)})^2<0$. Thus both of the above types of solitons are unstable for $p,q\gg1$. However, by the same argument as in the case with $\beta=2/3$, the core-symmetric soliton with a sufficiently low energy may still be stable.

Let us note that Eqs. (3.22) cannot be used for the stability analysis of the core-asymmetric soliton which exists in the narrow strip near the upper boundary in Fig. 2(b), because that strip has zero width in the limit $p,q\to\infty$ (cf. Sec. II). However, we believe that this type of soliton is stable because its shape must be very close to that of the two-component asymmetric soliton of the NLDC, which is known to be stable (see Sec. II).

$\beta=2$, $u_1u_2<0$, $v_1v_2<0$. For the core-symmetric soliton, all the three eigenvalues given by Eq. (3.20) are imaginary, while for AS1 and AS3 solitons, $(\lambda_1^{(1,-1)})^2<0$, see Eqs. (2.16a) and (2.16c). Thus these three types of solitons are unstable for $p,q\gg1$, with their instability growth rate being of the order $(1/\sqrt{\varepsilon})$. As we showed in Sec. IV, the AS2 soliton is also unstable; however, its instability growth rate is of the order $O(1)$, i.e., much less than that of the other solitons in this subcase. Note that the characteristic distance of the soliton evolution (such a distance is sometimes called a ''soliton period'' or dispersion length) for $p,q\gg1$ is of the order $O(1/\varepsilon)$, and therefore the AS2 soliton can propagate several soliton periods in the fiber before being destroyed by the instability. In fact, a situation in which a weakly unstable soliton could propagate a long distance before it decayed, due to a weak instability, into another, stable soliton state, has been recently reported [4] for the model of a weakly birefringent fiber.

Let us now consider some open questions regarding stability of the solitons of the DCDP. The first open question is rather obvious: Is the core-symmetric soliton in Fig. 2(a) stable in the limit $(p,q)\to\infty$? We believe that a better way to seek the answer to this question, rather than doing the next-order analytical calculations, would be to model the dynamics of the soliton in question numerically. In fact, solution of the other two problems formulated below will also require numerical simulations of Eqs. (1.1).

The second open problem concerns the boundary of soliton stability at finite (and low) energies. The numerical results for the NLDC [1] suggest that the instability of the core-symmetric solitons of the DCDP whose components have the same sign in both cores sets in right after the bifurcation curve where the (first) core-asymmetric soliton is created. However, for the core-symmetric solitons with $u_1u_2<0$ and/or $v_1v_2<0$, the situation may not be that simple. Indeed, a weak oscillatory instability of the antisymmetric (with $u_1=-u_2$) soliton of the NLDC sets in *before* that soliton undergoes a bifurcation in the parameter space. [Note that in that case, the bifurcation leads to the creation of a double-humped soliton; in our work, solitons of the latter type and thus the bifurcations that lead to their creation are excluded by the choice of ansatz (1.2).] Thus the stability of the core-symmetric solitons at finite energies requires a more thorough investigation. We would expect, however, that at very low energies, those solitons are stable, in analogy with the results for the NLDC.

Finally, another open problem concerns the final state to which the unstable solitons would evolve. In the case $\beta=2/3$, the answer to this appears to be clear, because there is only one stable soliton solution (for large energies), namely, the first core-asymmetric one in Fig. 1(a). However, for $\beta=2$, at least *two* solitons appear to be stable: these are the core-asymmetric soliton in Fig. 2(a) and the core-asymmetric soliton with small v components in Fig. 2(b). Also, further studies may reveal that the core-symmetric soliton in Fig. 2(a) is stable. Thus the final soliton state of the evolution of a pulse for $\beta=2$ could essentially depend on the initial conditions. We also speculate that in this case, one may have a long-term (quasi) periodic switching between the cores, similar to what one has in the two-component NLDC for sufficiently low energies, when both the symmetric and the antisymmetric solitons of that model are stable.

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